G Research IMO Problem 2

July 16, 2019

Problem

Let $g : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a function such that

$$g(1) + \cdots + g(n) \equiv 0 \pmod{n}.$$

Prove that there exist bijections $f_1, f_2 : \{1, \ldots, n\} \to \{1, \ldots n\}$ such that

$$f_1(x) + f_2(x) \equiv g(x) \pmod{n}$$

for all $x \in \{1, \ldots, n\}$.

Solution

(M. Hall Jr. (1952))

Let $f_1, f_2 : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be bijections, and let x_0 and x_1 be distinct points of $\{1, \ldots, n\}$. Given $y \in \{1, \ldots, n\}$, we will show how to modify f_1 and f_2 to bijections f_1' and f_2' so that

$$f_1'(x) + f_2'(x) \equiv f_1(x) + f_2(x)$$

for all $x \notin \{x_0, x_1\}$, and

$$f_1'(x_1) + f_2'(x_1) \equiv y.$$

By repeating this procedure we can find bijections f_1 and f_2 so that $f_1(x) + f_2(x) \equiv g(x)$ for all x except $x = 1$, say. But then also at $x = 1$ we will have

$$f_1(1) + f_2(1) \equiv -\sum_{x \neq 1}(f_1(x) + f_2(x)) \equiv -\sum_{x \neq 1} g(x) \equiv g(1).$$

Define x_2 by

$$f_1(x_0) + f_2(x_2) \equiv y.$$

Define $x_3, x_4, \ldots$ by requiring that

$$f_1(x_{j-1}) + f_2(x_{j+1}) \equiv f_1(x_j) + f_2(x_j)$$

for $j \geq 2$. Let $x_b = x_a$ $(a < b)$ be the first repetition in this sequence. We claim that $a \in \{0, 1\}$. Otherwise, we would have, for each $j \in \{a, \ldots, b-1\}$,

$$f_1(x_{j-1}) + f_2(x_{j+1}) \equiv f_1(x_j) + f_2(x_j),$$

or

$$f_1(x_{j-1}) - f_1(x_j) \equiv f_2(x_j) - f_2(x_{j+1}).$$

By summing these from $j = a$ to $j = b - 1$ we would get

$$f_1(x_{b-1}) - f_1(x_{a-1}) \equiv f_2(x_a) - f_2(x_b) \equiv 0,$$

whence $x_{b-1} = x_{a-1}$, contradicting our hypothesis that $x_b = x_a$ is the first repetition.

Now define f_1' and f_2' for $x \neq x_0$ by

$$\begin{aligned} f_1'(x_j) &= f_1(x_{j-1}) && (j \in \{1, \ldots, b-1\}) \\ f_1'(x) &= f_1(x) && (x \notin \{x_0, \ldots, x_{b-1}\}). \end{aligned}$$

and

$$\begin{aligned} f_2'(x_j) &= f_2(x_{j+1}) && (j \in \{1, \ldots, b-1\}) \\ f_2'(x) &= f_2(x) && (x \notin \{x_0, \ldots, x_{b-1}\}). \end{aligned}$$

Then it is easy to check that $f_1'(x_1)+f_2'(x_1) \equiv y$ and $f_1'(x)+f_2'(x) \equiv f_1(x)+f_2(x)$ for $x \neq x_0, x_1$. There is a unique way to define f_1' and f_2' on x_0 so that f_1' and f_2' are bijections. Specifically,

$$f_1'(x_0) = f_1'(x_{b-1})$$

and

$$f_2'(x_0) = \begin{cases} f_2(x_0) & \text{if } a = 1, \\ f_2(x_1) & \text{if } a = 0. \end{cases}$$