G Research IMO Challenge

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Challenge 1

Nim* is a game similar to Nim, with one difference: there is a single 'passing token' which either player may take in lieu of their turn if it is still available. Once either player takes the passing token the game plays exactly like a normal game of Nim. The game ends when there are no stones left, regardless of whether or not the passing token has been taken. The game is played under normal play convention, ie the last player to make a move wins.

The initial state contains the passing token in addition to piles of size 1, 3, 5, . . . , 2019.

Can the first player force a win? If so, how? If not, how can player two force a win?

Solution

Write all integers in binary. Define the Nim sum of two integers $a \oplus b$ to be the integer formed by bitwise-xor of the binary representation of a, b. It is well-known that in the game of Nim with piles $a_1, \ldots, a_n$, the first player wins iff $a_1 \oplus a_2 \oplus \cdots \oplus a_n = k \neq 0$. Suppose the most significant bit of k is at index i. The strategy is to find a_j whose bit at index i is 1, then remove tokens from pile a_j until there is $a_j \oplus k < a_j$ tokens left. The resulting Nim sum of all the piles is exactly 0. Note that on every turn, the Nim sum must change, thus if the second player has Nim sum 0, he will be forced to give player one a Nim sum that is not 0.

Group the piles into 3 groups G_1, G_2, G_3, with G_1 containing the single pile of size 1, G_2 containing the piles $\{3, 5, 7\}$, and G_3 containing the rest of the piles. Denote $\bigoplus G_i = \bigoplus_{a \in G_i} a$. Set $k_i = k_i(G_i) = \bigoplus G_i$ and $k = k_1 \oplus k_2 \oplus k_3$. We call a game state *good* if it satisfies one of the following:

1. The passing token is available, and $k_i = 1$ for exactly 2 distinct i.
2. The passing token is not available, and is a winning Nim game, i.e. $k \neq 0$.

We call a state *bad* if it satisfies one of the following:

1. The passing token is available, and $k_i = 1$ for all i.
2. The passing token is not available, and is a losing Nim game, i.e. $k = 0$.

Note that the bad and good states are disjoint. We claim that every good state has a move to a bad state, and every bad state only has moves to a good state. Thus a good state is a first player win. Our initial state has $k_1 = k_2 = 1, k_3 = 2$, so is a good state, hence is a first player win. We shall now prove the claim.

Suppose we have a good state. If the passing token is not available, then we are done by playing normal Nim. Otherwise, 2 of the k_i are 1. This means that there are still stones left, so the game has not ended. WLOG $k_1 = k_2 = 1$. If $k_3 = 0$, then take the passing token, resulting state is bad state 2. Otherwise, $k_3 > 1$. Suppose k_3 has most significant bit at index $i > 0$, so $2^i \leq k_3 < 2^{i+1}$. Find pile a in G_3 whose ith bit is 1. Then $b = a \oplus k_3 \oplus 1 < a$. Remove stones from a until b stones left. The resulting state is a bad state 1.

Suppose we have a bad state. If the passing token is not available, we are done by normal Nim. Otherwise, $k_1 = k_2 = k_3 = 1$. If we take the passing token, resulting state is a Nim game with $k = 1$, so is a good state 2. If we do not take the passing token, say we take stones from a pile in group G_1, then the resulting game has $k_1 \neq 1$ and $k_2 = k_3 = 1$, and we are in good state 1. Thus we have proven the claim.