PROBLEM 3

G RESEARCH

Problem

Find the largest value of K such that, for all $x, y, z \geq 0$, the following inequality holds:

$$\sum_{\text{cyc}} \left(x^2+1\right)^2 - 3 \geq \sum_{\text{cyc}} x^2 \left(Kx + (2\sqrt{2}-K)y\right)$$

Note that $\sum_{\text{cyc}}$ means the "cyclic sum" of an expression.
So $\sum_{\text{cyc}} x^2 = x^2 + y^2 + z^2$ and $\sum_{\text{cyc}} x^2 y = x^2 y + y^2 z + z^2 x$

Solution

This ended up being quite straightforward. If we substitute in $x = \sqrt{2}, y = z = 0$ we get that:

$$3^2 + 1 + 1 - 3 \geq 2\sqrt{2} \cdot K$$
$$\Leftrightarrow K \leq 2\sqrt{2}$$

But substituting in $K = 2\sqrt{2}$ gives:

$$\sum_{\text{cyc}} \left(x^4 + 2x^2\right) \geq \sum_{\text{cyc}} \left(2\sqrt{2}x^3\right)$$
$$\Leftrightarrow \sum_{\text{cyc}} \left(x^4 - 2\sqrt{2}x^3 + 2x^2\right) \geq 0$$
$$\Leftrightarrow \sum_{\text{cyc}} x^2 \cdot \left(x - \sqrt{2}\right)^2 \geq 0$$